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ON SEARCHING FOR EVENTS OF LIMITED DURATION

by

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Abstract

An observer wishes to detect as many as possible of a set of events. The events arise at several discrete points according to independent Poisson processes, and the lifetimes of individual occurrences are independent and identically distributed random variables. The specific problem is: given that the observer can only "visit" one point per unit time, in what sequence should he make his "visits" so as to maximize the steady-state fraction of events he detects? Some results about the optimal search policy are obtained, and the best policy is found precisely in some circumstances.

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FOREWORD

The research project, "Innovative Resource Planning in Urban Public Safety Systems," is a multidisciplinary activity, supported by the National Science Foundation, and involving faculty and students from the M.I.T. Schools of Engineering Science, Architecture and Urban Planning, and Management. The administrative home for the project is the M.I.T. Operations Research Center. The research focuses on three areas: 1) evaluation criteria, 2) analytical tools, and 3) impacts upon traditional methods, standards, roles, and operating procedures. The work reported in this document is associated primarily with category 2, in which a set of analytical and simulation models are developed that should be useful as planning, research, and management tools for planners and decision-makers in many agencies.

Richard C. Larson
Principal Investigator

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Introduction

There is considerable variety in the problems on search theory that have appeared in the literature. Some of the objects of search are assumed fixed in location[1] while others are moving on trajectories governed by random parameters.[2] In some cases decoys can frustrate the search effort;[3] other times the object sought is a human being consciously attempting to evade the pursuers.[4,5] Yet common to many of these problems are the assumptions that only one object is being sought, and that there are no absolute time limits on the search except those that might implicitly arise in the constraints on available effort.

In practice, however, one might wish to detect as many as possible of a set of events which arise randomly in time, can occur simultaneously, and are of limited duration. A major example involves the preventive patrol activities of police departments designed to intercept crimes in progress. Larson[6] has applied traditional search theory to this problem to obtain useful results when the probability of more than one crime at any moment is small. In this paper, we consider a relatively simple problem on optimal search for events generated in random fashion over time (possibly at a high rate) at several points. While the model used is not itself greatly applicable to practical problems, it might have some value in indicating both the possibilities and limitations of analytical work in situations of this kind.

The Problem

Events are assumed generated at N discrete points $(1,\ldots,N)$ according to independent Poisson processes with fixed parameters $(q_1,\ldots,q_N)$. The durations of the events are independent and identically distributed random variables with cumulative distribution function $F(x)$. There is no limitation on the number of simultaneous occurrences at any one point. At instants of time exactly one unit apart, an investigator visits one and only one of the points. He detects all events in progress at a location at the instant of his visit. The question is: what search strategy should he employ to maximize the steady-state fraction of events he observed at least once? An equivalent formulation would have him maximize the average number of events sighted per unit time; it is under this criterion that we examine the problem.

Two Points

We consider first the special case of exactly two points (1 and 2) with "event generation" parameters q_1 and q_2; we assume q_1 exceeds q_2. Suppose a_k is the expected number of events seen for the first time when the observer returns to point 1 after an absence of exactly k units of time. Then a_k satisfies:

$$a_k = \int_0^k q_1(1 - F(t))dt$$

a_k is clearly a nondecreasing function of K. Because $F(t)$ is nondecreasing in t, the quantity $d_k = a_{k+1} - a_k$ is nonincreasing,

meaning that a concavity property characterizes the set of a_k's. The comparable quantity b_k for point 2 is of course given by $b_k = Aa_k$ where $A = q_2/q_1$.

What we want to do is specify the pattern of visits to 1 and 2 which gives the highest "rate of return" in terms of the relevant a_k's and b_k's. It is fairly clear that the strategy should not be biased toward the slower point 2; we illustrate simply the general style of argumentation in this problem by the proof of the rather unsurprising remark below.

Remark: If q_1 exceeds q_2, it is never advantageous to visit point 2 two or more times in a row.

Proof: We show first that one need never visit point 2 exactly twice in a row. Suppose the optimal policy includes exactly two consecutive visits to 2 at least once. Then the itinerary 1 2 2 1 X must arise over some 5-unit period, where X can be either 1 or 2.

a) Suppose X = 1. Then the average number of sightings in the last three units of this period is $b_1 + a_3 + a_1$. If, however, one switched the times of the second visits to 1 and 2 to yield the sequence 1 2 1 2 1, the average number of detections in the last three units would change to $2a_2 + b_2$. Now $(2a_2 + b_2) - (b_1 + a_3 + a_1) = (a_2 - a_1) - (a_3 - a_2) + A(a_2 - a_1) = (1 + A)(a_2 - a_1) - (a_3 - a_2) \geq 0$, since $d_k = a_{k+1} - a_k$ is nonincreasing in k and $A > 0$. Coupled with the fact that expected gain over the first two units of the period is unchanged, this means that the alteration has not weakened the policy, and thus that the double-visit to 2 was at best unnecessary.

b) Suppose X = 2. Now the average number of sightings in the last three units is $b_1 + a_3 + b_2$. If the second visit to 2 is replaced by a visit to 1 (i.e., sequence becomes 1 2 1 1 2), the number of events observed in the last three units takes an average value of $a_2 + a_1 + b_3$. Now $(a_2 + a_1 + b_3) - (b_1 + a_3 + b_2) = (1 - A)(a_1 + a_2 - a_2) = (1 - A)(a_1 - d_2) \geq 0$ since $d_2 \leq a_1$. Thus once again the alteration has improved the policy, except in the special case F(t) = 0 for t < 3 when one can do just as well without consecutive visits to 2.

Using similar arguments, one can establish that it is never advantageous to visit point 2 three or more times in a row. The proof consists in demonstrating that for any $k \geq 3$, one can at least match a policy calling for k successive visits to 2 by changing the second of these k trips to a trip to 1; we omit the details here.

In the two point case, we can think of the observer's travel pattern as a series of cycles from point 2 back to itself. What we have shown is that one can always achieve the highest possible detection rate under a policy in which every cycle time is at least two. Over a cycle of time k, the expected number of events detected per unit time is $\dfrac{b_k + a_2 + (k - 2)a_1}{k} \equiv e_k$ (where only the final return to 2 is counted toward the current cycle). It is clear that, except in the "degenerate" case when two values of k yield exactly the same average gain, there is one cycle length that should be used repeatedly to achieve the

optimal detection rate. This optimal c must satisfy the relations:

$$\frac{b_c + a_2 + (c - 2)a_1}{c} > \frac{b_{c-1} + a_a + (c - 3)a_1}{c - 1}$$

and " $$> \frac{b_{c+1} + a_2 + (c - 1)a_1}{c + 1}.$$

(A simple argument based on concavity in the a_k's and b_k's establishes the concavity of the e_k's, and thus assures the uniqueness of a c satisfying both inequalities.) This c follows:

$$c = (\max k \mid 2a_1 - a_2 > kb_{k-1} - (k - 1)b_k).$$

Summarizing, we have:

<u>Theorem 1</u>

An optimal search policy for two points is a regular cyclical pattern under which the observer visits the slower point every c units of time and spends all other units at the busier point. The value of c is given by $c = (\max k \mid 2a_1 - a_2 > kb_{k-1} - (k - 1)b_k)$. The expected number of events observed per unit time under this policy is

$$\frac{b_c + a_2 + (c - 2)a_1}{c}.$$

This result corresponds to one's intuitive expectation. We should note that the remark proved earlier could have been obviated in the proof of the theorem by the simple observation that a policy with 2-2 cycles of length 1 is necessarily inferior to one with cycles of length 2; we proved the remark only for illustrative purposes, as noted. In the next section, we move on to situations with several discrete points.

Many Points

Suppose that now events are generated at N points designated (1,...,N). The first thing we will do is show that, for our purposes, we can restrict our attention to cyclic policies.

Theorem

If all events have bounded duration, there exists an optimal search policy that is cyclic.

Proof: A search policy S is specified by a sequence of numbers $\{s_k\}$ for k = 0,1,2,... , where s_k is the point visited at time k. For S, we define the N-tuple $(a_{1k},\ldots,a_{Nk})$ as follows: at time k, let a_{jk} be the maximum possible age of an event then occuring at point j that has not been detected. If the observer is visiting point j at time k, $a_{jk} = 0$; otherwise, a_{jk} is the smallest of the following three quantities: 1) the time since the process began, 2) the time since the last visit to point j and 3) t, the maximum duration of any event (which is assumed finite). There must be some integer M such that $a_{jk} \leq M$ for all $j \leq N$ and all k; clearly $M \leq \hat{t}$, where $\hat{t}$ is the smallest integer not less than t. Note that because the visits occur at unit intervals, any a_{jk} can be only an integer from 0 to M or the number t. Thus, as k varies from 0 to ∞, a_{jk} can take on at most M + 1 different values, and correspondingly the N-tuple takes on no more than $(M + 1)^N$ values over the entire evolution of the process. It is clear, then, that there must be some particular N-tuple, P, that arises at one time and reappears infinitely often.

The crucial point is that since the event generation parameters and event duration distribution are unchanging with time, the N-tuple at any instant contains all the information from the past relevant to the average "rate of return" under any policy in the future. Now suppose S is an optimal search policy. Let t_j be the jth instant when the N-tuple equals P under policy S; also let g_j be the expected gain per unit time between (slightly after) t_j and (slightly after) t_{j+1} under the strategy and $g = \max_j g_j$. Since S is an optimal strategy, g is the upper limit on the overall detection rate one can achieve. But since g is the expected rate of detection between t_x and t_{x+1} for some x, one can obtain this average rate over the infinite time horizon by a simple expedient: just apply the policy S search-sequence between t_x and $t_{x+1} - t_x$ (except perhaps for an "edge-effect" period at the beginning of the process); since it yields the optimal "rate of return" the theorem is proved.

As a practical matter, the restriction that all events have durations below some finite upper bound is no restriction at all. Unfortunately, the theorem is of limited usefulness since we have no indication what the optimal cycle length is. The next result is more directly valuable, for it identifies some points the observer need not visit at all.

The "Exclusion" Theorem

Suppose that h_k is the expected number of new sightings at point H after a k-unit absence, and that $h_\infty = \lim_{k\to\infty} h_k$. Then to maximize the expected number of events detected per unit time, one should never visit H if $h_\infty < 2a_1 - a_2$. (a_k and d_k are as defined earlier; q_1 is still assumed the largest of the generation rates.)

Proof: Suppose that the best policy included at least one visit to H, which is m units since the last visit to busiest point 1 and n units before the next scheduled visit there. Then the expected number of events sighted in the visit to H and the next trip to 1 is at most $h_\infty + a_{m+n}$. If the visit to point H were replaced by another trip to 1, the average number of observations in the two time units changes to $a_m + a_n$.

Now, writing $a_m = a_1 + d_1 + \ldots + d_{m-1}$, $a = a_1 + d_1 + \ldots + d_{n-1}$ and $a_{m+n} = a_1 + d_1 + \ldots + d_{m+n-1}$, the upper bound on change in expected gain $(a_m + a_n) - (h_\infty + a_{m+n})$ follows:

$$(a_m + a_n) - (h_\infty + a_{m+n}) = a_1 + (d_2 - d_m) + \ldots$$

$$+ (d_{n-1} - d_{m+n-2}) - d_{m+n-1} - h_\infty \quad .$$

Since the d_k's are nonincreasing in k, the quantities in parentheses are nonnegative and $d_{m+n} \leq d_1$. Therefore,

$$a_m + a_n - a_{m+n} - h_\infty \geq a_1 - d_1 - h_\infty = 2a_1 - a_2 - h_\infty \; .$$

Hence if $h_\infty < 2a_1 - a_2$, the average gain has increased because of the policy alteration, implying that the policy involving the visit to H was not optimal, which proves the theorem.

The theorem gives us a basis for excluding some points from possible visits at once. But, except in the case of exactly two points, the requirement $h_\infty > 2a_1 - a_2$, while a necessary

condition for visiting P, is not a sufficient one. This is shown by the three-point situation $q_1 = 1$, $q_2 = 1-\varepsilon$, $q_3 = \varepsilon$, where $\varepsilon > 0$ but is very small. If $F(t) = \begin{matrix} 0 \text{ for } t < 2 \\ 1 \text{ for } t \geq 2, \end{matrix}$ then $2a_1 - a_2 = 0$ and $h_\infty = 2\varepsilon$. Yet despite the fact that $h_\infty > 2a_1 - a_2$, the best strategy calls for alternating between 1 and 2 and never visiting 3.

The "exclusion" theorem tells us which points we can ignore at once and, if only two points are not excluded, the problem is solved. But we have to consider what to do if more than two points remain. That is the subject of the balance of the paper.

Coexistence and Interference

Some implications of the concavity property are important in the forthcoming discussion. If the observer's visits to any given point have a mean time spacing of s, his mean detection rate there is easily shown to be highest when the intervals between successive visits vary as little as possible around s. Thus, if s is an integer, these intervals ideally should be exactly s, where by "ideally" we mean best for the point considered in isolation (although not necessary for the whole set of points among which the observer must "compromise"). If $s = k + y$ where k is an integer and $0 < y < 1$, the $k + 1$ and $1 - y$ at k. Now we are ready to prove a "coexistence" theorem.

The "Coexistence" Theorem

Suppose that for each of the N discrete points besides the busiest one (still 1), the best two-point search policy for points 1 and k is found. Let w_k be the time between two consecutive visits to point k (k > 1) under this policy. Suppose one can devise an N-point search policy under which

1) For each k, the observer visits point k every w_k units (k > 1).
2) Each visit to k is immediately preceded by and followed by a visit to 1.

Then the strategy just described is the optimal N-stage policy.

Proof: For notational ease, we deal with exactly three points: the extension to more points is immediate. Both w_2 and w_3 are assumed finite; otherwise the problem is trivial. Consider a strategy under which trips to 2 and 3 have average spacing s and v respectively. All other trips are to (busiest) point 1 so the mean spacing γ there follows

$$\gamma = (1 - \frac{1}{s} - \frac{1}{v})^{-1} .$$

From the discussion immediately preceding this theorem, an upper limit on the overall detection rate is

$$\frac{\bar{b}_s}{s} + \frac{\bar{c}_v}{v} + \frac{\bar{a}_\gamma}{\gamma}$$

where $b_k \equiv$ expected no. of sightings at 2 after a k-unit absence and $\bar{b}_s \equiv (1 - y)b_k + yb_{k+1}$; where s = k + y, k an integer $0 \leq y < 1$, etc.

The theorem's requirement that each visit to 2 and 3 both precede and follow a trip to 1 implies that both w_2 and w_3 must be at least 4. To compare the "coexistence" policy with all others, we divide other policies into two groups based on their s and v values.

a) Suppose $1/s + 1/v \leq 1/2$ for a particular strategy. Then $1 \leq \gamma \leq 2$ so, since γ can be written as $1 + \frac{\frac{1}{s} + \frac{1}{v}}{1 - \frac{1}{s} - \frac{1}{v}}$, use of the concavity property at point 1 allows us to express the upper limit on expected sighting rate as:

$$\frac{\bar{b}_s}{s} + \frac{\bar{c}_v}{v} + a_2(\frac{1}{s} + \frac{1}{v}) + a_1(1 - \frac{2}{s} - \frac{2}{v})$$

which can be rewritten as

$$(\frac{\bar{b}_s + a_2 + (s-2)a_1}{s}) + (\frac{\bar{c}_v + a_2 + (v-2)a_1}{v}) - a_1 .$$

Now the first quantity in parenthesis is the best detection rate under a two-point strategy for 1 and 2 with mean cycle time s. But this quantity is maximized when $s = w_2$; likewise, $v = w_3$ maximized the second quantity. Thus the "coexistence" policy, when it exists, is the best of all policies for which $1/s + 1/v \leq 1/2$, (i.e., at least half the visits are to the busiest point).

b) $1/2 < 1/s + 1/v \leq 2/3$*

In this situation, γ falls between 2 and 3. Using arguments analogous to those of part (a), we obtain as an upper limit on the average sighting rate the quantity

$$\left(\frac{\bar{b}_s + 2a_3 + (s-3)a_2}{s}\right) + \left(\frac{c_v + 2a_3 + (v-3)a_2}{v}\right) - a_3 \ . \qquad (1)$$

At least one, and possibly both, of s and v falls between 2** and 4. We consider below the case when both of them do; the argument for the case when one of s and v exceeds 4 is not very different.

If we write $s = k + y$ where $k = 2$ or 3 and $0 < y \leq 1$, the first quantity in parenthesis can be expressed as:

$$\frac{(1-y)b_k + yb_{k+1} + 2a_3 + (s-3)a_2}{s} =$$

$$(b_{k+1} - b_k + a_2) + \frac{2a_3 - 3a_2 + (k-1)b_k - kb_{k-1}}{s} \ . \qquad (2)$$

Since we know w_2 is at least 4, our Theorem 1 about two-point best policies implies that $(k+1)b_k - kb_{k+1} < 2a_1 - a_2$.

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* The upper limit 2/3 arises because under any policy with $1/s + 1/v > 2/3$ the busiest point is not the one visited most frequently. Clearly the best overall strategy cannot have this property, so demonstrating that coexistence is best whenever $1/s + 1/v \leq 2/3$ is sufficient for our purposes.

** The case $s = 2$ is not directly considered in this discussion but the extension to this case is straightforward.

Hence the numerator in the fraction on the right of (2) is at most $2a_1 + 2a_3 - 4a_2$, which is itself nonpositive because of concavity. Thus $\frac{2a_1 + 2a_3 - 4a_2}{k+y}$ can not decline as y is increased to 1, and therefore the s-related term in (1) is bounded by the value of that term when $s = 3$ or when $s = 4$ depending on k's value. When $s = 3$, the relevant part of (1) becomes $(b_3 + 2a_3)/3$; when $s = 4$, it is $(b_4 + 2a_3 + a_2)/4$. The latter (yes, concavity again) is the larger expression, and thus $(b_4 + 2a_3 + a_2)/4$ bounds the "s-component" of (1) for all $2 < s \leq 4$.

Identically, for v between 2 and 4, the second quantity in parenthesis has an upper bound of $(c_4 + 2a_3 + a_2)/4$, and thus the total sum in (1) is bounded from above by

$\frac{b_4 + c_4 + 2a_2}{4}$. But this quantity is exactly the upper limit which arose in part (a) for $s = 1/4$ and $v = 1/4$; since we showed in (a) that the "coexistence" observation rate is at least that large, our work is complete.

We call this a "coexistence" theorem because fairly stringent compatability conditions on w_2 and w_3 must hold for the theorem to be relevant. In particular, both w_2 and w_3 must have a common factor at least 4. In the ranges $2 \leq w_2 \leq 25$ and $2 \leq w_3 \leq 25$, for instance, this condition is satisfied for about 12% of the value pairs. When there are four or more points to be searched, the rarity of "coexistence" increases sharply.

What does one do when "coexistence" does not occur? It would be pleasant to find some more general theorem of which the results we have obtained are just special cases. But the author has been unable to locate one, and is in fact doubtful that such a theorem exists. However, common sense indicates a few approximation procedures one might use when the w_k's are not compatable in the necessary way and the total number of points for potential search is relatively small. We restrict our attention now to the case of exactly three points.

Suppose both w_2 and w_3 are at least 4 but have no common factor at least 4. (We ignore smaller values of w_2 and w_3 since simple trial-and-error methods are feasible in those cases.) One might approximate the best policy by the better of the two "adjacent coexistence policies" formed when each of the two numbers w_2 and w_3 is (separately) altered the minimum possible amount to be in coexistence with the other. (e.g., if $w_2 = 5$ and $w_3 = 8$, the two coexistence policies use $\bar{w}_2 = 4$, $w_3 = 8$ and $w_2 = 5$, $\bar{w}_3 = 10$.) Or, one might instead attempt to implement the two two-point policies with 1 as closely as possible (the "interference" policy); when "collisions" arise between the times of indicated visits to 2 and 3, one resolves them in favor of 2 or 3 depending on how $b(w_2)^* + c(2w_3)$ compares to $b(2w_2) + c(w_3)$. (Note that under this approach, the intervalbetween consecutive visits to 1 will sometimes be 3.) Since, as we have seen, an upper bound on expected "gain" is

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* $b(w_2) = b_{w_2}$; this is just a notational change to avoid a double-subscript.

$$\frac{b(w_2) + a_2 + (w_2 - 2)a_1}{w_2} + \frac{c(w_3) + a_2 + (w_3 - 2)a_1}{w_3} - a_1$$

one knows the maximum possible loss associated with any particular estimation. We illustrate these approached with a simple example.

<u>Numerical Example</u>

Suppose $q_1 = 1$, $q_2 = .057$, $q_3 = .024$ and $F(t) = 1 - \exp(.1t)$ $t \geq 0$. The values of a_k for $k = 1,\ldots,12$ are given below; note that $b_k = .057a_k$ and $c_k = .024a_k$.

<u>k</u>	<u>a_k</u>
1	.95
2	1.82
3	2.59
4	3.30
5	3.94
6	4.52
7	5.04
8	5.51
9	5.94
10	6.33
11	6.68
12	6.99

where $F(t) = 1 - \exp(.1t)$

Since $b_\infty = .57$, $c_\infty = .24$ and $a_2 - 2a_1 = .08$, both 2 and 3 are candidates for search. From Theorem 1, $w_2 = (\max k \mid kb_{k-1} - (k-1)b_k < 2a_1 - a_2)$; here $w_2 = 7$ and, similarly, $w_3 = 11$. No coexistence. We thus consider the approximate policies discussed.

1) Adjacent Coexistence Policy 1: $\bar{w}_2 = 11$, $w_3 = 11$.

$$\text{Average gain } \bar{g} = \frac{c_{11} + b_{11} + 2a_2 + 7a_1}{11} = .986.$$

2) Adjacent Coexistence Policy 2: $w_2 = 7$, $\bar{w}_3 = 7$.

$$\bar{g} = \frac{c_7 + b_7 + 2a_2 + 3a_1}{7} = .985$$

3) Interference Policy ("collisions" resolved in favor of point 2)

$$\bar{g} = \frac{5c_{11} + c_{22} + 11b_7 + 13a_2 + 2a_1 + 45a_1}{77} = .985.$$

4) Upper limit on g:

$$M = \frac{b_7 + a_2 + 5a_1}{7} + \frac{c_{11} + a_2 + 9a_1}{11} - a_1 = .987.$$

From (4) we see that our approximations did extremely well. There is no particular significance to the fact that coexistence triumphed over interference here; under the duration distribution function for which $a_3 = 2.65$ and the other a_k's are unchanged (note that the concavity requirement is still satisfied), interference is the best of the three approximations. The fact that coexistence estimates may or may not be better than those from a very different approach suggests the difficulty of finding a unifying theorem to cover all circumstances.

Conclusions

We have considered a sequential-search problem involving the detection of as many as possible of a series of events, events that are random in their times and places of origin and in their durations. We solved the problem exactly in the case of exactly two generation points and showed that, more generally,

there is a cyclic optimal search policy. Then we proved "exclusion" and "coexistence" theorems that sometimes allow the reduction of problems with many points to a series of two-point problems. We also discussed some simple approximation methods.

As noted, our results are useful primarily when the total number of points where events arise is relatively small. Thus an obvious area for further investigation is large-scale problems with the same underlying model. Another possible direction for further research involves the alteration and/or generalization of the assumptions we used. While it is unclear how far one can progress in these areas using analytical methods, the successes we did achieve hint that further effort might indeed be fruitful.

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