STATIONARITY AND INVERTIBILITY REGIONS
FOR LOW ORDER STARMA MODELS

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Summary

The building of STARMA, space-time autoregressive moving average, models requires a working knowledge of the conditions under which a particular model represents a stationary process. Constraints on the parameter space that ensure stationarity are developed for all STARMA models of autoregressive temporal order less than or equal to two and spatial order less than or equal to one when the model form utilizes scaled weights. Invertibility conditions for these same models are also given.

Keywords:

STARMA Stationarity Conditions

STARMA Invertibility Conditions

STARMA Modeling

1. Introduction

The STARMA, space-time autoregressive moving average, model family has proven useful in modeling time histories of spatially located data (1,2,3,7). This model family utilizes the spatial autocorrelation usually exhibited by space-time series to provide an efficient representation of the system.

The basic mechanism for this representation is a hierarchical spatial ordering of the neighbors of each site and a sequence of weighting matrices, $W^{(\ell)}$. Matrix $W^{(\ell)}$ has elements $w_{ij}^{(\ell)}$ that are nonzero if and only if sites i and j are ℓ^{th} order neighbors, and $W^{(0)}$ is the identity matrix.

The STARMA family of models takes the form

$$\underset{\sim}{z}(t) = \sum_{k=1}^{p} \sum_{\ell=0}^{\lambda_k} \phi_{k\ell} W^{(\ell)} \underset{\sim}{z}(t-k)$$

$$- \sum_{k=1}^{q} \sum_{\ell=0}^{m_k} \theta_{k\ell} W^{(\ell)} \underset{\sim}{\varepsilon}(t-k) + \underset{\sim}{\varepsilon}(t) \qquad [1]$$

where

$\underset{\sim}{z}(t)$ is the Nx1 vector of observations at the N locations in the system at time t

$\underset{\sim}{\varepsilon}(t)$ is the vector of unexplained residuals at time t

p is the autoregressive order

q is the moving average order

λ_k is the spatial order of the k^{th} autoregressive term

m_k is the spatial order of the k^{th} moving average term

$\left.\begin{array}{l}\phi_{k\ell}\\ \theta_{k\ell}\end{array}\right\}$ are parameters

$$E[\underset{\sim}{\varepsilon}(t)] = \underset{\sim}{0}$$

$$E[\underset{\sim}{\varepsilon}(t)\ \underset{\sim}{\varepsilon}(t+s)'] = \begin{cases} G & s=0 \\ 0 & \text{otherwise} \end{cases}$$

This model is referred to as a STARMA$\left(p_{\lambda_1,\lambda_2,\ldots\lambda_p}, q_{m_1,m_2,\ldots m_q}\right)$ model.

Two special subclasses of the STARMA model are of note. When q=0, only autoregressive terms remain and hence, the model class carries the label space-time autoregressive or STAR model. Models that contain no autoregressive terms (p=0) are referred to as STMA models.

The purpose of this paper is to investigate the conditions under which the process generated by [1] is a stationary stochastic process. In model building, the selection of a unique representation of a given space-time system requires restriction of the parameter space to be equivalent to the stationary domain. That is, a covariance structure does not uniquely identify a particular model form unless the model parameters are required to fall in the stationary region.

Section 2 defines stationarity of the general multivariate ARMA model, of which the STARMA is a special case. Section 3 examines the stationarity of the STAR(2_{11}) model, a second order autoregressive model of the first spatial order. Stationarity conditions on the 4 parameters of this model are developed for the case of a general scaled $W^{(1)}$ matrix. A scaled weighting matrix is one in which $\sum_{j=1}^{n} w_{ij}^{(\ell)} = 1$ for all i. From the stationarity region of the STAR(2_{11}), the corresponding regions for the STAR(2_{10}), STAR($2_{0,0}$) and STAR(1_1) follow directly. These regions are presented and displayed.

Whereas all moving average models are stationary (the stationarity of a model is a function only of the autoregressive terms), one usually requires that the moving average terms be such that the model is invertible. Section 4 discusses invertibility of the STMA model.

2. Stationarity and the General Multivariate ARMA Process

A discrete-time vector process, $\underset{\sim}{z}(t)$, is called stationary if for all n and h, the distributions of $\underset{\sim}{z}(t_1),\underset{\sim}{z}(t_2),\ldots\underset{\sim}{z}(t_n)$ and $\underset{\sim}{z}(t_1+h),\underset{\sim}{z}(t_2+h),\ldots\underset{\sim}{z}(t_n+h)$ are the same. If the $\underset{\sim}{z}(t)$ have finite mean square, this means that

$$E[\underset{\sim}{z}(t)\ \underset{\sim}{z}(t+s)'] = E[\underset{\sim}{z}(t+h)\ \underset{\sim}{z}(t+h+s)'] = \Gamma(s)$$

and the covariance between observations s time lags apart does not change with time.

Consider the vector process generated by the general ARMA model

$$\underset{\sim}{z}(t) - \sum_{j=1}^{p} B(j)\ \underset{\sim}{z}(t-j) = \underset{\sim}{\varepsilon}(t) - \sum_{k=1}^{q} A(k)\ \underset{\sim}{\varepsilon}(t-k)\ , \qquad [2]$$

where the B(j) and A(k) are NxN square parameter matrices, p is the autoregressive order of the process, q is the moving average order and the $\underset{\sim}{\varepsilon}(t)$ are random shocks with

$$E[\underset{\sim}{\varepsilon}(t)\ \underset{\sim}{\varepsilon}(t+s)'] = \begin{cases} G & s=0 \\ 0 & \text{otherwise} \end{cases}$$

It is clear from this formulation that $E[\underset{\sim}{z}(t)] = \underset{\sim}{0}$.

The conditions for stationarity of this general multivariate model can be found in (4) and are repeated here without proof. If every x_u that solves

$$\det\left[x_u^{\,p}\, I - \sum_{j=1}^{p} B(j)\ x_u^{\,p-j}\right] = 0 \qquad [3]$$

lies inside the unit circle ($|x_u| < 1$), then the vector process [2] will be stationary.

3. Stationarity Conditions for the STAR Model

It is clear that since the STARMA model is a constrained version of the general multivariate ARMA model, the stationary conditions for the general model can be used to test for stationarity of the STARMA. In particular we will address the stationarity of the STAR(2_{11}) model and derive constraints on the parameters of this model that will ensure stationarity conditions for simpler models can be determined by setting to zero certain parameter(s) of the STAR(2_{11}) as we shall see.

The STAR(2_{11}) model

$$\underset{\sim}{z}(t) = (\phi_{10}I + \phi_{11}W^{(1)})\underset{\sim}{z}(t-1) + (\phi_{20}I + \phi_{21}W^{(1)})\underset{\sim}{z}(t-2) + \underset{\sim}{\varepsilon}(t) \quad [4]$$

is therefore stationary if every x_u that solves

$$\det\left[x_u^2 I - (\phi_{10}I + \phi_{11}W^{(1)})x_u - (\phi_{20}I + \phi_{21}W^{(1)})\right] = 0$$

has $|x_u| < 1$. This is equivalent to requiring that every x_u solving

$$\det\left[(\phi_{11}x_u + \phi_{21})W^{(1)} - (x_u^2 - \phi_{10}x_u - \phi_{20})I\right] = 0 \quad [5]$$

lies inside the unit circle. Letting $\lambda(x_u) = (x_u^2 - \phi_{10}x_u - \phi_{20})$, it is easily seen that $\lambda(x_u)$ are the eigenvalues of the matrix $C = (\phi_{11}x_u + \phi_{21})W^{(1)}$.

It is known cf. (5) that if λ is an eigenvalue of the square matrix $C = \{c_{ij}\}$, then λ lies within or on the boundary of at least one of the N circles in the complex plane defined by

$$|\lambda - c_{ii}| = \sum_{j \neq i} |c_{ij}|.$$

For the case at hand $C = (\phi_{11}x_u + \phi_{21})W^{(1)}$ and since $w_{ii}^{(1)} = 0$ and $\sum_{i=1}^{N} w_{ij}^{(1)} = 1$ for all i, these N circles are all identical to

$$|\lambda(x_u)| = |\phi_{11}x_u + \phi_{21}|.$$

Thus, substituting for $\lambda(x_u)$, we have that every x_u solving [5] necessarily satisfies

$$|x_u^2 - \phi_{10}x_u - \phi_{11}| \leq |\phi_{11}x_u + \phi_{21}| \quad [6]$$

Our task is now one of finding the largest set $S \subset R^4$ such that if $(\phi_{10},\phi_{11},\phi_{20},\phi_{21}) \in S$ then every x_u satisfying [6] also has $|x_u| < 1$. Any set with this property will be a stationarity region for the STAR(2_{11}) model since it has been shown that all solutions to [5] also satisfy [6]. Requiring all x_u satisfying [6] to lie inside the unit circle will therefore ensure stationarity of [2].

Since we are interested in ensuring that every x that solves $|x^2 - \phi_{10}x - \phi_{20}| \leq |\phi_{11}x + \phi_{21}|$ has $|x| < 1$, consider the nonlinear programming problem

maximize x

subject to

$$|x^2 - \phi_{10}x - \phi_{20}| \leq |\phi_{11}x + \phi_{21}|$$

Substituting x = a + bi, where a and b are now real numbers we have

maximize $(a^2 + b^2)^{1/2}$

subject to

$$\left[(a^2 - b^2 - \phi_{10}a - \phi_{20})^2 + (2ab - \phi_{10}b)^2\right]^{1/2} \leq \left[(\phi_{11}a + \phi_{21})^2 + (\phi_{11}b)^2\right]^{1/2}$$

Upon squaring both sides of the constraint, multiplying and rearranging, we arrive at an equivalent problem

maximize $(a^2 + b^2)^{\frac{1}{2}}$

subject to $g(a,b) \leq 0$ where

$$g(a,b) = a^4 + b^4 + 2a^2b^2 - 2\phi_{10}a^3 - 2\phi_{10}ab^2 + (\phi_{10}{}^2 - \phi_{11}{}^2 - 2\phi_{20})a^2$$

$$+ (\phi_{10}{}^2 - \phi_{11}{}^2 + 2\phi_{20})b^2 + 2(\phi_{10}\phi_{20} - \phi_{11}\phi_{21})a + \phi_{20}{}^2 - \phi_{21}{}^2.$$

We will now show that if (a^*,b^*) is an optimal solution to this problem, then it is necessarily true that $g(a^*,b^*) = 0$. Suppose we have an optimal solution (a^*,b^*) that satisfies $g(a^*,b^*) < 0$. Then (a^*,b^*) is also an optimal solution to the unconstrained problem maximize $(a^2 + b^2)^{\frac{1}{2}}$. But obviously the optimal solution to the unconstrained problem is unbounded. Therefore any optimal solution to the problem maximize $(a^2 + b^2)^{\frac{1}{2}}$ subject to $g(a,b) \leq 0$ must satisfy $g(a,b) = 0$.

We will now procede to find a set S such that if $(\phi_{10},\phi_{11},\phi_{20},\phi_{21}) \in S$, then every solution to

$$|x_u{}^2 - \phi_{10}x_u - \phi_{20}| = |\phi_{11}x_u + \phi_{21}| \tag{7}$$

lies inside the unit circle. From the above arguments, every solution to [6] will also lie inside the unit circle, and S will be a stationarity region.

There are four solutions to [7]:

$$x_1 = \frac{(\phi_{10} + \phi_{11}) + \sqrt{(\phi_{10} + \phi_{11})^2 + 4(\phi_{20} + \phi_{21})}}{2}$$

$$x_2 = \frac{(\phi_{10} + \phi_{11}) - \sqrt{(\phi_{10} + \phi_{11})^2 + 4(\phi_{20} + \phi_{21})}}{2}$$

$$x_3 = \frac{(\phi_{10} - \phi_{11}) + \sqrt{(\phi_{10} - \phi_{11})^2 + 4(\phi_{20} - \phi_{21})}}{2}$$

$$x_4 = \frac{(\phi_{10} - \phi_{11}) - \sqrt{(\phi_{10} - \phi_{11})^2 + 4(\phi_{20} - \phi_{21})}}{2}$$

corresponding to the solutions of the two equations $x_u^2 - \phi_{10}x_u - \phi_{20} = \phi_{11}x_u + \phi_{21}$ and $x_u^2 - \phi_{10}x_u - \phi_{20} = -\phi_{11}x_u - \phi_{21}$. We now use the result that

$$\left|\frac{y - \sqrt{y^2 - 4v}}{2}\right| < 1 \text{ and } \left|\frac{y + \sqrt{y^2 - 4v}}{2}\right| < 1 \text{ if and only if}$$

$$|y| < 1 + v$$

$$\text{and} \quad v < 1$$

to construct the set S that ensures $|x_u| < 1$ $u = 1,2,3,4$. The resulting constraints on $(\phi_{10},\phi_{11},\phi_{20},\phi_{21})$ are:

$$-\phi_{20} - \phi_{21} < 1$$

$$|\phi_{10} + \phi_{11}| < 1 - \phi_{20} - \phi_{21}$$

$$-\phi_{20} + \phi_{21} < 1$$

$$|\phi_{10} - \phi_{11}| < 1 - \phi_{20} + \phi_{21}$$

which can be rewritten as

$$-\phi_{20} + |\phi_{21}| < 1$$

$$|\phi_{10} + \phi_{11}| < 1 - \phi_{20} - \phi_{21} \qquad [8]$$

$$|\phi_{10} - \phi_{11}| < 1 - \phi_{20} + \phi_{21}$$

Constraint set [8] serves to define the set S of $(\phi_{10},\phi_{11},\phi_{20},\phi_{21})$ values that ensure stationarity of the STAR(2_{11}) model.

Stationarity regions for other low order STAR models can be found by setting to zero certain parameter(s) in constraint set [8]. In particular, the stationarity region for the STAR(2_{10}) model is found by setting $\phi_{21} = 0$ and combining to give:

$$\begin{aligned} -1 &< \phi_{20} \\ |\phi_{10}| + |\phi_{11}| &< 1 - \phi_{20} \end{aligned} \qquad [9]$$

This region is pictured in Figure 1.

The stationarity region for the STAR(2_{00}) results when ϕ_{11} and ϕ_{21} are zero. The resulting region

$$\begin{aligned} -1 &< \phi_{20} \\ |\phi_{10}| &< 1 - \phi_{20} \end{aligned} \qquad [10]$$

is seen to be identical to the stationarity region of the univariate AR(2) region (Figure 2).

In a similar manner, the stationarity region for the two parameter STAR(1_1) model can be found by setting ϕ_{20} and ϕ_{21} equal to zero giving

$$|\phi_{10}| + |\phi_{11}| < 1. \qquad [11]$$

This region is shown in Figure 3. Finally the STAR(1_0) model has a single stationarity constraint $|\phi_{10}| < 1$, identical to the AR(1) condition.

4. Invertibility Regions of the STMA Model

It can be seen from the general form of the STARMA model [1] that there is a duality between the autoregressive and moving average terms of the STARMA. The autoregressive terms of the model modify the observations in exactly the same way that the moving average terms modify the residuals.

We have just developed constraints on the autoregressive parameters of the STAR model that ensure a stationary model. The observations from a stationary process can be expressed as a weighted linear combination of current and past residuals, and furthermore these weights will converge to zero as the time lag increases. It should be clear, then, that the stationarity constraints applied to the moving average parameters will ensure that $\underset{\sim}{\varepsilon}(t)$ can be expressed as a weighted linear combination of current and past errors with weights converging to zero. This property is called invertibility, and consequently the invertibility regions for various low order STMA models are simply the stationarity regions of the STAR with θ_{ij} replacing ϕ_{ij}.

5. Conclusions

The stationary and invertibility regions are developed for those models from the STARMA model class with temporal order less than or equal to two and spatial order less than or equal to one. These regions apply to the STARMA model using general scaled weights.

6. Acknowledgements

This work was performed under Grant Nos. 75NI-99-091 and 78NI-AX-004 of the National Institute of Law Enforcement and Criminal Justice. Points of view or opinions stated in this document are those of the authors and do not necessarily represent the official position or policies of the United States Department of Justice.

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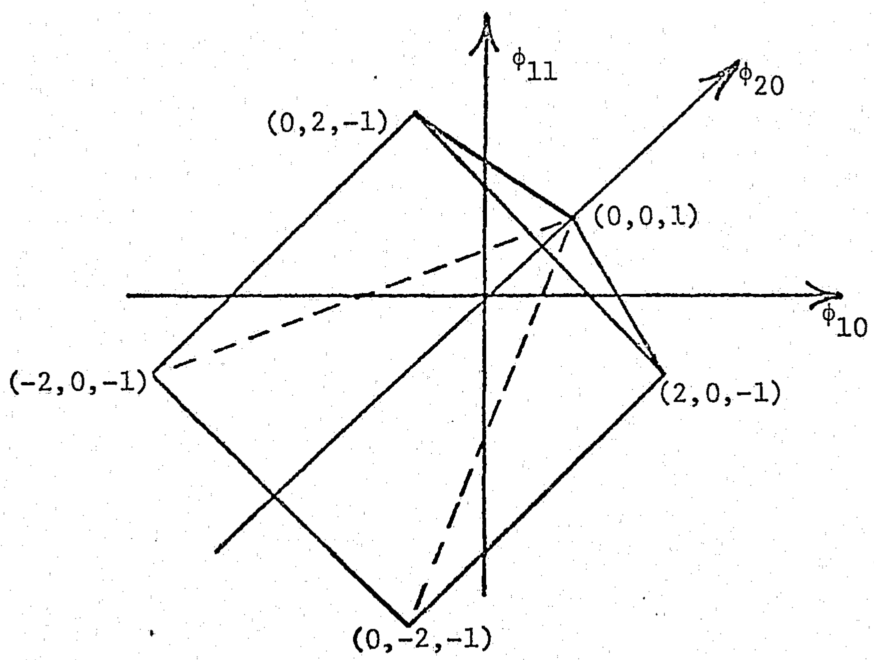

Figure 1a. The Stationarity Region of the STAR(2_{10})

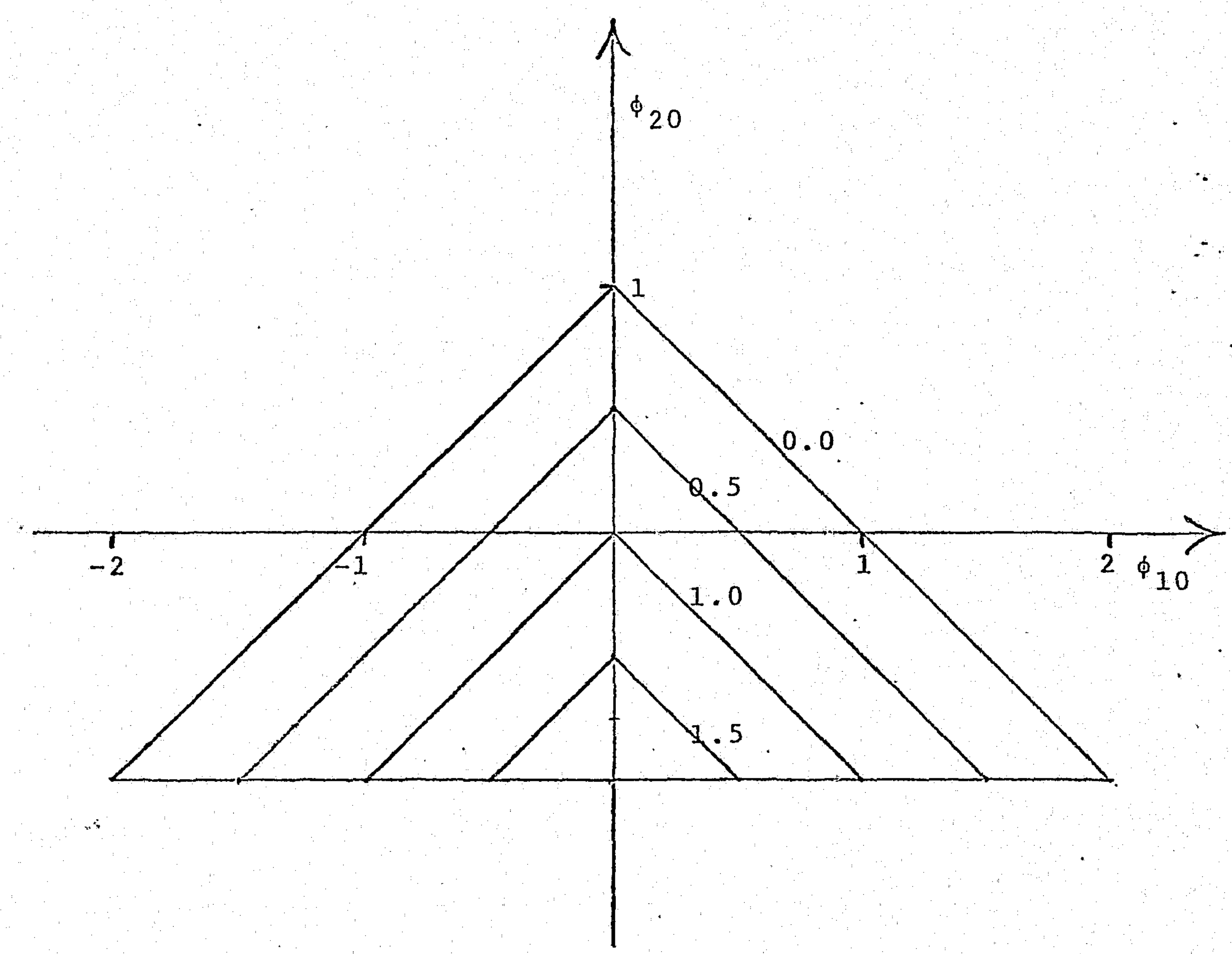

Figure 1b. Stationarity Contours for the STAR(2_{10}) Model at Various Values of $|\phi_{11}|$

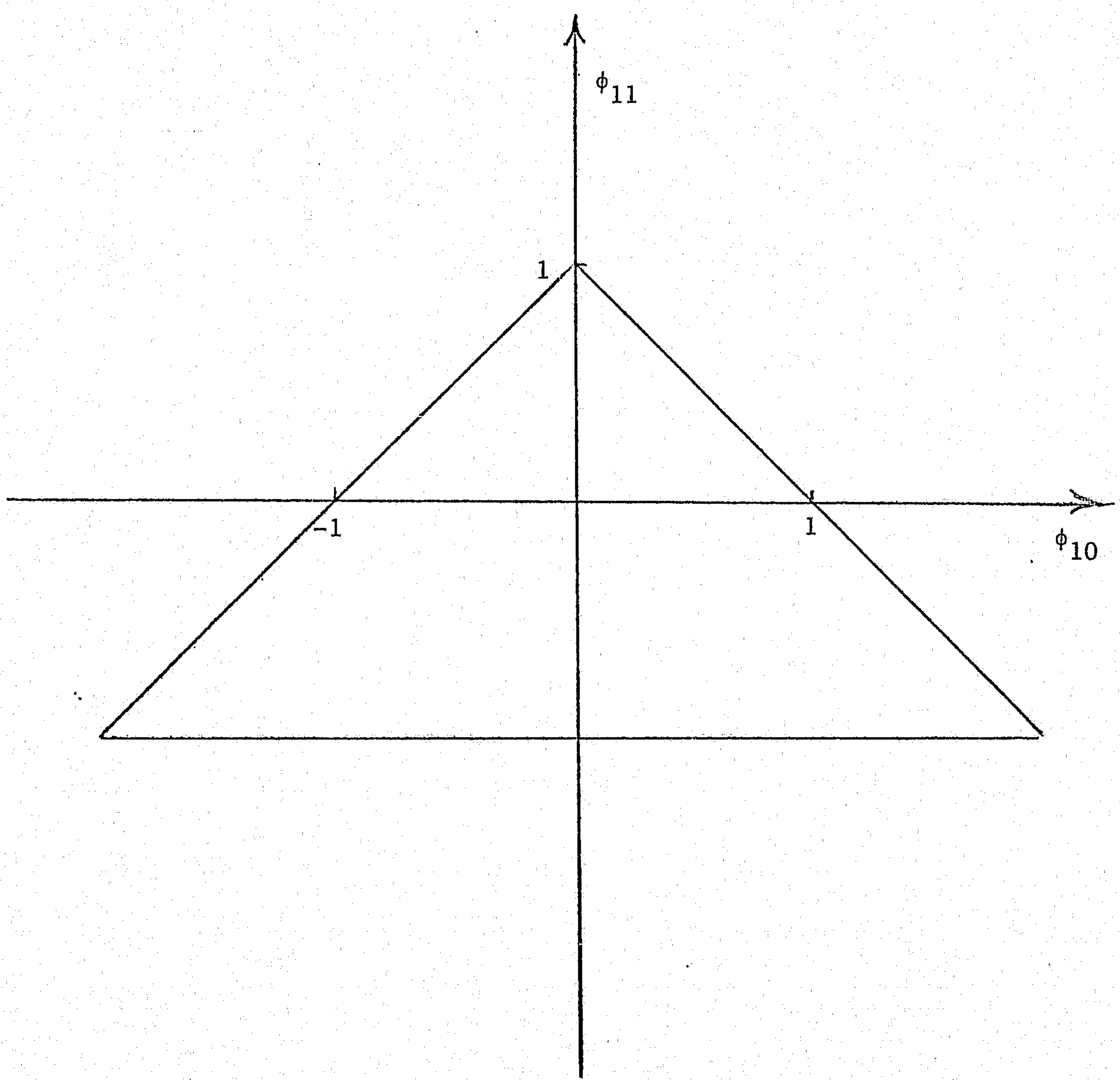

Figure 2. The Stationarity Region of the STAR(2_{00}) Model

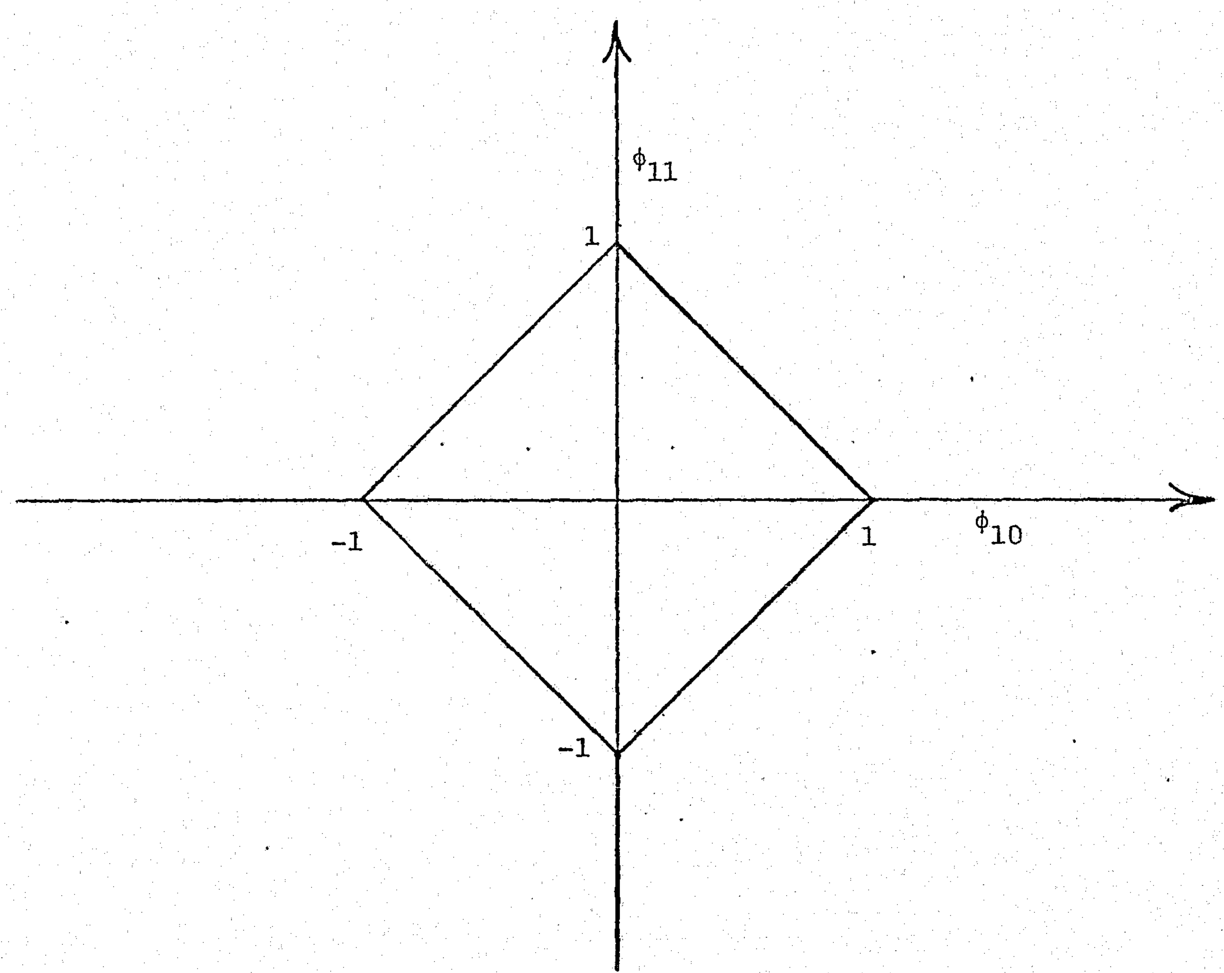

Figure 3. The Stationarity Region of the STAR(1_1) Model

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