Industrial and Systems Engineering
Report Series No. J-80-13
September, 1980

BUILDING EMPIRICAL SPACE-TIME MODELS FOR FORECASTING, INTERVENTION ANALYSIS AND CONTROL OF UNKNOWN SYSTEM MECHANISMS

by

Stuart Jay Deutsch *

and

Susumu Fujii **

* School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA

** Faculty of Engineering, Kobe University, Kobe, JAPAN

BUILDING EMPIRICAL SPACE-TIME MODELS FOR FORECASTING, INTERVENTION ANALYSIS AND CONTROL OF UNKNOWN SYSTEM MECHANISMS

by

Stuart Jay Deutsch
Professor
School of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332 USA

and

Susumu Fujii
Associate Professor
The Faculty of Engineering
Kobe University
Nada, Kobe 657 Japan

BUILDING EMPIRICAL SPACE-TIME MODELS FOR FORECASTING, INTERVENTION ANALYSIS AND CONTROL OF UNKNOWN SYSTEM MECHANISMS

by

Stuart Jay Deutsch

and

Susumu Fujii

ABSTRACT

The building of empirical models for space-time systems via identification procedures for unknown system mechanisms is described. The space-time models of the autoregressive moving average form (STARMA) are displayed in various system control-related considerations and problems such as forecasting, intervention analysis and transfer function modeling.

I. INTRODUCTION

A flexible class of empirical models is the multiplicative autoregressive moving average family. These models along with the model building procedure commonly referred to as the Box-Jenkins method [1] have proven very useful in a wide spectrum of statistical analyses which have focused on system description [2,3,4], forecasting [2,5,6], intervention analysis [7,8,9,10,11] and process control [1,12,13,14].

Since these models are univariate they are only applicable to a single stream of time series data. Recently the univariate time series model class and the three-stage model building procedure has been extended to space-time systems by Pfeifer and Deutsch [15,16, 17,18,19,20,21,22,23]. These space-time autoregressive integrated moving average models (STARIMA models) describe N series, where each series represents the pattern from one of N regions, along with the interrelationships between the regions. The use of the space-time models in forecasting [24,25] and intervention analyses [26,27] has been described.

The purpose of this paper is to describe the use of space-time models in developing empirical input-output models of unknown systems for process control. It should be noted that, in these cases the underlying structure of the transfer function for the system is unknown but must be determined from system data by appropriate

identification and estimation procedures. In the following section an overview of the STARIMA model class and the three-stage model building procedure is presented. The adoption of these models for preliminary process control through forecasting and intervention analysis is described in section three. In the last section, the procedures for extending STARIMA modeling capability to building the STARIMA transfer function model from which control schemes can be described are presented.

II. THE SPACE-TIME AUTOREGRESSIVE MOVING AVERAGE MODEL

The STARMA model class is characterized by linear dependence lagged in both space and time. Assume that observations $z_i(t)$ of the random variable $Z_i(t)$ are available at each of N fixed locations in space (i=1,2,...N) over T time periods. The N locations in space will be referred to as sites and can represent a variety of situations. The autoregressive form of the space-time model would express the observation at time t and site i, $z_i(t)$ as a linear combination of past observations at zone i and neighboring zones. If the same relationship holds for every site in the system, the process is said to exhibit spatial stationarity and is thus amenable to these forms of space-time models.

To assist in the formulation of this space-time model, the following definition of the spatial lag operator is needed. Let $L^{(\ell)}$, the spatial lag operator of spatial order ℓ, be such that

$$L^{(0)}\ z_i(t) = z_i(t)$$

$$L^{(\ell)}\ z_i(t) = \sum_{j=1}^{N} w_{ij}^{(\ell)}\ z_j(t)$$

where $w_{ij}^{(\ell)}$ are a set of weights with

$$\sum_{j=1}^{N} w_{ij}^{(\ell)} = 1$$

for all i and $w_{ij}^{(\ell)}$ nonzero only if sites i and j are ℓ^{th} order neighbors. The matrix representation of the set of weights $w_{ij}^{(\ell)}$ is $W^{(\ell)}$, an N by N square matrix with each row summing to one. If $\underset{\sim}{z}(t)$ is an (Nx1) column vector of the observations $z_i(t)$, i = 1,2, ...N, then

$$L^{(0)}\ \underset{\sim}{z}(t) = W^{(0)}\ \underset{\sim}{z}(t) = I_N\ \underset{\sim}{z}(t)$$

and

$$L^{(\ell)}\ \underset{\sim}{z}(t) = W^{(\ell)}\ \underset{\sim}{z}(t) \quad \text{for } \ell > 0$$

The specification of the form of weights $w_{ij}^{(\ell)}$ for various positive ℓ's is a matter left up to the model builder who may choose weights to reflect the configuration. The $w_{ij}^{(\ell)}$ may be chosen to reflect physical properties of the observed system such as the length of the common boundary of the distance between contiguous sites i and j.

These weights, however, must reflect a hierarchical ordering of spatial neighbors. First order neighbors are those "closest" to the site of interest. Second order neighbors should be "farther" away than first order neighbors, but "closer" than third order neighbors. Figure 1 shows the first four spatial order neighbors of a particular site for both a two-dimensional grid system and a one-dimensional line of sites. This definition

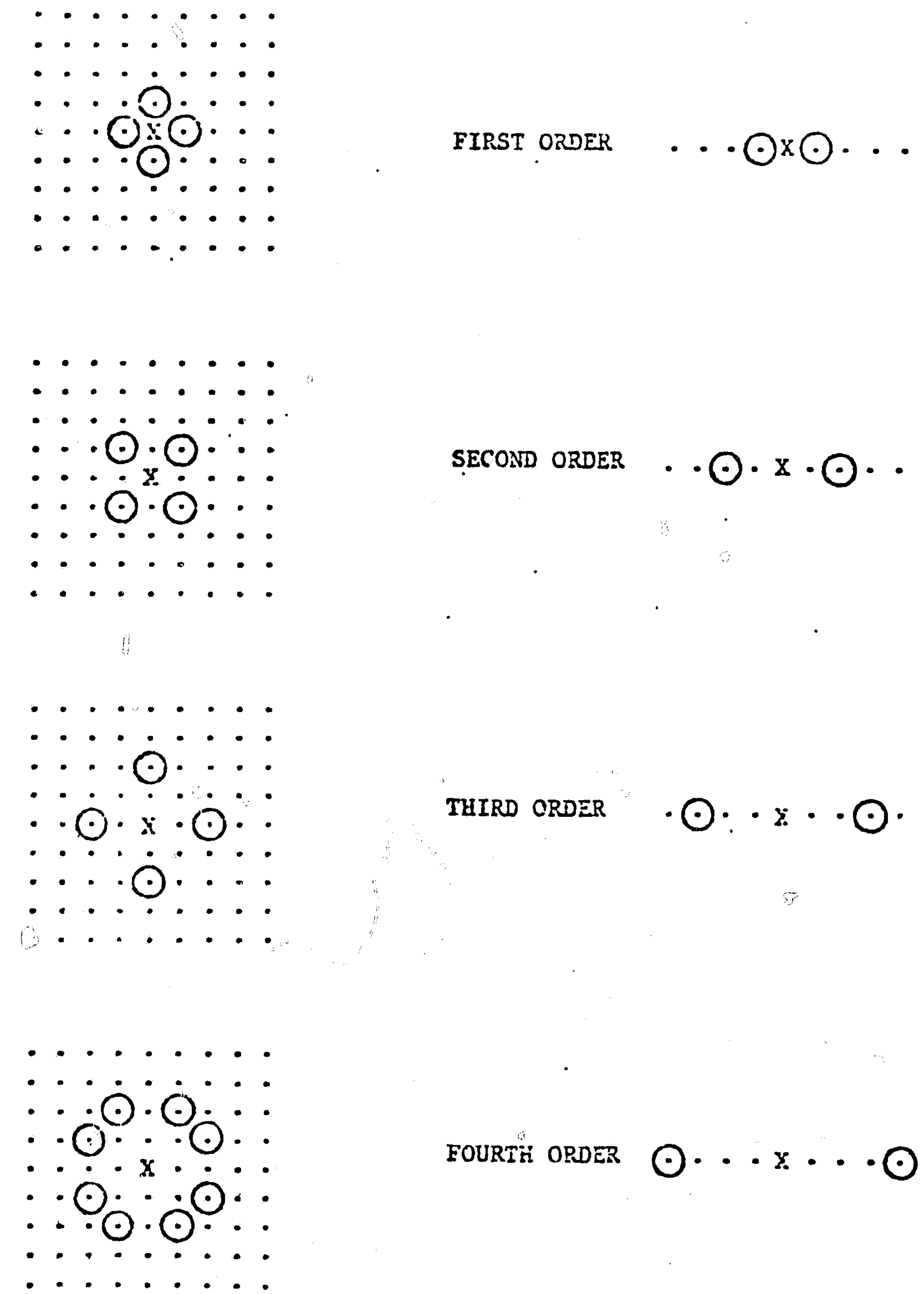

FIGURE 1

Spatial Order in Two and One Dimensional Systems of spatial order represents an ordering in terms of euclidean distance of all sites surrounding the location of interest.

With this definition of spatial order in hand, we are now ready to present the STARMA model. Analogous to univariate time series, $z_i(t)$ will be expressed as a linear combination of past observations and errors. Here, however, instead of allowing dependence of $z_i(t)$ only with past observations and errors at site i, dependence is allowed with neighboring sites of various spatial order. In particular

$$z_i(t) = \sum_{k=1}^{p} \sum_{\ell=0}^{\lambda_k} \phi_{k\ell} L^{(\ell)} z_i(t-k) - \sum_{k=1}^{q} \sum_{\ell=0}^{m_k} \theta_{k\ell} L^{(\ell)} \varepsilon_i(t-k) + \varepsilon_i(t) \qquad (1)$$

where

p is the autoregressive order

q is the moving-average order

λ_k is the spatial order of the k^{th} autoregressive term

m_k is the spatial order of the k^{th} moving-average term

$\left.\begin{matrix}\phi_{k\ell}\\ \theta_{k\ell}\end{matrix}\right\}$ are parameters

and the $\varepsilon_i(t)$ are random normal errors with

$$E\,[\varepsilon_i(t)] = 0$$

$$E\,[\varepsilon_i(t)\varepsilon_j(t+s)] = \begin{cases}\sigma^2 & i=j,\ s=0\\ 0 & \text{otherwise}\end{cases}$$

This model is referred to as a STARMA $\left(p_{\lambda_1,\lambda_2,\ldots\lambda_p},\ q_{m_1,m_2,\ldots m_q}\right)$ model.

The same model in vector form is

$$\underset{\sim}{z}(t) = \sum_{k=1}^{p} \sum_{\ell=0}^{\lambda_k} \phi_{k\ell} W^{(\ell)} \underset{\sim}{z}(t-k) - \sum_{k=1}^{q} \sum_{\ell=0}^{m_k} \theta_{k\ell} W^{(\ell)} \underset{\sim}{\varepsilon}(t-k) + \underset{\sim}{\varepsilon}(t) \qquad (2)$$

with $\underset{\sim}{\varepsilon}(t)$ normal with mean zero

and $E\,[\underset{\sim}{\varepsilon}(t)\underset{\sim}{\varepsilon}(t+s)'] = \begin{cases}\sigma^2 I_N & s=0\\ 0 & \text{otherwise}\end{cases}$

Given observations from a space-time system, the model building problem is the selection and analysis of an appropriate form of model from the general STARIMA class. The model building procedure, then, is the method by which the experimental data leads to a particular model form and particular parameter values from the total family of STARIMA models. It is a three-stage iterative procedure of identification, estimation and diagnostic checking.

Identification is the first stage of the model building process. In STARIMA model building, the primary tools in identification are the space-time autocorrelation and space-time partial autocorrelation functions [15,18,20,23].

In a manner completely analogous to that of univariate time series, STARIMA processes are each characterized by a distinct space-time partial and autocorrelation function. Whereas univariate autoregressive models exhibit autocorrelation functions that decay exponentially with time and partial correlation functions that cut off after p lags, the STAR process exhibits a space-time correlation function that trails off with both space and time and partial autocorrelations that cut off after p lags in time and λ_p lags in space. Similarly, univariate moving average models have just the opposite, autocorrelations that cut off after q lags and partials that decay

over time. The STMA$\left(q_{m_1},\ldots,m_q\right)$ model similarly is characterized by an autocorrelation function that cuts off after q temporal lags and m_q spatial lags and partials that tail off spatially and temporally. Mixed models exhibit partials and autocorrelations that both tail off. In the univariate case, they tail off only in time, whereas space-time mixed ARMA processes have space-time autocorrelation functions that decay both with time and space. These general characteristics form the basis of the identification stage. Further discussion of the identification considerations is contained in [18].

After a tentative model from the STARMA model family has been chosen by comparison of sample to theoretical autocorrelation patterns in the identification phase, it is necessary to estimate the parameters. The best estimates of the $\hat{\underset{\sim}{\phi}}$ and $\hat{\underset{\sim}{\theta}}$ from many points of view are the maximum likelihood estimates, but because of the start-up difficulties associated with time series estimation, conditional maximum likelihood (CML) is usually employed. A comparison of alternate estimation procedures on the accuracy of estimates obtained is presented in [16].

The diagnostic checking stage examines the residuals from the fitted model. If the fitted model adequately represents the data, these residuals should be white noise, i.e., should be distributed normally with mean zero and variance-covariance matrix equal to $\sigma^2 I_N$ with all autocovariances at non-zero lags equal to 0.

Various tests are available for testing the residuals for white noise. Probably the most useful test (especially in the context of the three-stage modeling procedure for space-time models) is that of calculating the sample space-time autocorrelations of the residuals and checking for additional significance structure. In [20,23] the standardization for the residual space-time autocorrelation to allow hypothesis tests for significance was developed. If the residuals are not random the pattern is identified and the tentative model updated.

III. PROCESS CONTROL

As with the univariate time series models the STARIMA models can be used to project past observed occurrences in developing future forecasts. The forecasts from these models are best in a statistical sense because they possess the property of minimum mean square error. A complete description of the forecasting of these models and the properties of the forecasts are contained in [24]. It should be emphasized that the forecasts produced from the models will be accurate representations of future events if in fact the system continues to operate in the future as it has in the past.

An implicit assessment of the process state from historical tendencies can be made by comparing future forecasts made at time T to the corresponding observed process states monitored at T+k, k=1,2,... . Thus, probability estimates can be made of the likelihood of the current observation deviating from the forecast given the underlying process, as manifest by the historical data, is unchanged [5].

Explicit measurement of change in the historical process is also possible [7,9,10]. To test whether a process has shifted or changed in the mean level after time T and to estimate the magnitude of this shift the form of the historical model prior to T can be augmented to allow for a shift parameter. Statistical estimation and hypothesis tests can then be directly applied in evaluating whether a real shift or change has been observed. The coupling of this type analysis with new activities in the system and their time frame of implementation is what we shall refer to generally as intervention analysis. The procedures for selected univariate ARIMA models were presented originally by Box and Tiao [7].

Parallel considerations for the STARIMA models have recently been developed [27]. In this case, we have N time series data at N locations, and some event which occurs after the t=n observation but before n_1+1 observations potentially causes the process to go from in control to out of control, a shift in operating level. The model for monitoring this process is,

for the pre-intervention data component $(t \leq n_1)$:

$$z_i(t) - \mu_i = \sum_{k=1}^{p} \sum_{\ell=0}^{\lambda_k} \phi_{k\ell} \left[\sum_{j=1}^{N} w_{ij}^{(\ell)} (z_j(t-k) - \mu_j) \right] - \sum_{k=1}^{q} \sum_{\ell=0}^{m_k} \theta_{k\ell} \left[\sum_{j=1}^{N} w_{ij}^{(\ell)} \varepsilon_j(t-k) \right] + \varepsilon_i(t)$$

$t = 1,2,\ldots n; \quad i = 1,2,\ldots N$

for the post-intervention data component $(t > n_1)$:

$$z_i(t) - (\mu_i + \delta_i) = \sum_{k=1}^{t-(n_1+1)} \sum_{\ell=0}^{\lambda_k} \phi_{k\ell} \left[\sum_{j=1}^{N} w_{ij}^{(\ell)} (z_j(t-k) - (\mu_j + \delta_j)) \right] + \sum_{k=t-n_1}^{p} \sum_{\ell=0}^{\lambda_k} \phi_{k\ell} \left[\sum_{j=1}^{N} w_{ij}^{(\ell)} (z_j(t-k) - \mu_j) \right] - \sum_{k=1}^{q} \sum_{\ell=0}^{m_k} \theta_{k\ell} \left[\sum_{j=1}^{N} w_{ij}^{(\ell)} \varepsilon_j(t-k) \right] + \varepsilon_j(t)$$

$t = n_1+1, n_1+2, \ldots n_1+n_2; \quad i = 1,2,\ldots N .$

This model is referred to as the Intervention STARIMA and is denoted by STARIMA$\left(p_{\lambda_1,\ldots\lambda_p}; q_{m_1,\ldots m_q}\right)_I$.

The intervention analysis models described are similar in function to quality control charts such as the Shewart $\bar{X}$ chart [28] in that effectively a process model is developed for an attribute and future realizations of the process are looked at with regard to whether they fall within the control limits that are derived from the form of the process model. For univariate series the types of changes that have occurred in the process level can be modeled by the dynamic intervention models [8]. Here a dummy variable (0,1) denoting absence or presence of a potential intervention is used to develop a transfer function. In this form, the output $\underset{\sim}{Z}(t)$ related to the absence or presence of a planned intervention by a dynamic model $\underset{\sim}{D}(t)$ describing the observed impulse response function plus an additive noise component at the output, $\underset{\sim}{N}(t)$. Thus $\underset{\sim}{Z}(t)=\underset{\sim}{D}(t)+\underset{\sim}{N}(t)$

where $\underset{\sim}{N}(t)$ follows the appropriate model form from the ARIMA model class. The dynamic intervention model is a special case of the empirical models of single stochastic input and output illustrated in a model building framework by Box and Jenkins [1].

Whereas both approaches develop a transfer function model for the control of an unknown system, the latter is particularly germane to physical systems in which the relationship between the input and output is unknown but the identity of the input variable is known and is controllable on-line. In dynamic intervention models the input-output relationship is unknown as is the specific named input variable which may be one of many variables associated with an "intervention" program and therefore collectively represented by a proxy variable denoted by its on or off state, 1 or 0 respectively.

IV. BUILDING A SPACE-TIME MODEL FOR CONTROL

In this section, the extension of the univariate control models to accommodate space-time processes and the corresponding identification and model building procedures for these unknown systems (both with regard to transfer function mechanism and input variable specification) are presented.

The unknown transfer function system to be modeled is represented by paired realizations $(z_i(t), x_i(t))$ of random variables $(Z_i(t), X_i(t))$ available at each of N fixed locations in space $(i=1,2,\ldots,N)$ over T time periods. The $\underset{\sim}{x}(t)$ represent input streams

and the $\underset{\sim}{z}(t)$ output streams. At the output of the black box with unknown transfer function is an additive noise component $\underset{\sim}{N}(t)$ whose underlying model form is also unkown. ~~Figure 2 illustrates this system.~~ For simplicity a line system of inputs and outputs is illustrated. In general the configuration of the inputs or outputs may take any form of spatial configuration or map. The model form to be developed from data for this system is,

$$\underset{\sim}{z}(t) = V\underset{\sim}{x}(t-b) + \underset{\sim}{N}(t) ,$$

where V is an NxN matrix of impluse response weights with the $(\ell,k)^{th}$ elements of the form $\sum_s \nu_{s,\ell k}B^s$, and b is the delay at the i^{th} location.

The procedures for the determination of the space-time transfer function model are:

1) Using the identification, estimation and diagnostic checking procedures (overviewed in section II), build the appropriate space-time model for the input streams, $\underset{\sim}{x}(t)$.

2) Prewhiten the output streams $\underset{\sim}{z}(t)$ by the space-time structural model of the outputs to obtain the prewhitened outputs $\underset{\sim}{\xi}(t)$ where

$$\left[\underset{\sim}{x}(t) - \sum_{k=1}^{p}\sum_{\ell=0}^{\lambda_k} \phi_{k\ell}\, W^{(\ell)}\, \underset{\sim}{x}(t-k) + \sum_{k=1}^{q}\sum_{\ell=0}^{m_k} \theta_{k\ell}\, W^{(\ell)}\, \underset{\sim}{\xi}(t-k)\right]\underset{\sim}{z}_t = \underset{\sim}{\xi}(t) .$$

3) Compute the sample space-time cross-correlation function of the prewhitened outputs $\underset{\sim}{\xi}(t)$ and the input residuals $\underset{\sim}{\varepsilon}(t)$, $\hat{\rho}_{s,\ell k}$ where,

$$\hat{\rho}_{s,\ell k} = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T-s} L^{(\ell)} \xi_i(t) \, L^{(k)} \varepsilon_i(t+s)}{\left[\sum_{i=1}^{N} \sum_{t=1}^{T} (L^{(\ell)} \xi_i(t))^2 \cdot \sum_{i=1}^{N} \sum_{t=1}^{T} (L^{(k)} \varepsilon_i(t))^2 \right]^{1/2}} \qquad s=0,\pm 1,\pm 2,\ldots$$

to obtain estimates of the $(\ell,k)^{th}$ impulse response weights $\hat{v}_{s,\ell k}$ where,

$$\hat{v}_{s,\ell k} = \frac{\hat{\rho}_{s,\ell k} \, \hat{\sigma}_k}{\hat{\sigma}_\ell} \qquad s=0,\pm 1, \pm 2,\ldots .$$

4) Compute an estimate of the additive noise streams $\hat{\underset{\sim}{N}}(t)$ from

$$\hat{\underset{\sim}{N}}(t) = \underset{\sim}{z}(t) - V\underset{\sim}{x}(t-b) .$$

5) Build the appropriate STARIMA model for the space-time noise process $\hat{\underset{\sim}{N}}(t)$ to obtain residuals $\hat{\underset{\sim}{a}}(t)$.

6) Simultaneously refit the obtained combined model,

$$\underset{\sim}{z}(t) = V\underset{\sim}{x}(t-b) + \underset{\sim}{N}(t) ,$$

and diagnostically check the residuals $\hat{\underset{\sim}{a}}(t)$ to be random normal errors with

$$E(\underset{\sim}{a}(t) \, \underset{\sim}{a}(t+s)') = \begin{cases} \sigma^2 I_n & s=0 \\ 0 & \text{otherwise} \end{cases}$$

and the input and residual streams to be uncorrelated,

$$E(\underset{\sim}{x}(t) \, \underset{\sim}{a}(t+s)') = \underline{0} .$$

From the space-time transfer function model the design of feedforward and/or feedback control schemes can proceed. Details

of the design of control equations from univariate models are contained in [1]. These principles could also be adapted to the space-time model formulations. In the case where the specific input streams are unknown by name and all "intervention" attributes are collectively designated by absence or presence (0,1) in the space-time transfer function model the feedback control scheme based upon deviation from desired target value of the outputs becomes a means of evaluating the collective intervention as to whether to continue the program.

V. CONCLUSIONS

An overview of model building procedures for space-time systems that use data structure to identify a statistically appropriate form was presented. These STARIMA models were adapted to the control-related considerations of forecasting; intervention analysis, in which changes in a space-time process are monitored to determine whether the process is still "in control;" and space-time transfer function models for direct use in feedforward and/or feedback control.

VI. ACKNOWLEDGEMENT

This work was performed under Grant No. 78-NI-AX-0040 from the National Institute of Justice. Points of view or opinions stated herein are those of the authors and do not necessarily represent the official position or policies of the United States Department of Justice.

REFERENCES

1. BOX, G. E. P. and JENKINS, G. M. (1976). Time Series Analysis: Forecasting and Control, rev. ed. San Francisco: Holden-Day.
2. DEUTSCH, S. J. (1978). Stochastic models of crime rates. International Journal of Comparative and Applied Criminal Justice 2, 127-151.
3. DEUTSCH, S. J. and OGELSBY, G. B. (1979). Analysis of the toxic effects of physical and chemical properties of hypolimnetic waters by time series. Environment International 2, No. 3, 133-138.
4. DEUTSCH, S. J. and WU, S. M. (1974). Analysis of wear during grinding by empirical-stochastic models. Wear 29, 247-257.
5. DEUTSCH, S. J. (1978). Deterrence effectiveness measurement. Criminology 16, No. 1, 115-131.
6. DEUTSCH, S. J. and PFEIFER, P. E. (1980). Normative forecasting; application of stochastic models. International Journal of Comparative and Applied Criminal Justice 4, No. 2.
7. BOX, G. E. P. and TIAO, G. (1965). A change in the level of a non-stationary time series. Biometrika 52, 181-192.
8. BOX, G. E. P. and TIAO, G. (1975). Intervention analysis with applications to economic and environmental problems. Journal of the American Statistical Association 70, 70-92.
9. DEUTSCH, S. J. and ALT, F. B. (1976). Estimation of shifts in stochastic models of crime occurrence. Proceedings of the 7th Annual Conference on Modeling and Simulation, Pittsburgh.
10. DEUTSCH, S. J. and ALT, F. B. (1977). The effect of Massachusetts' gun control law on gun-related crimes in the city of Boston. Evaluation Quarterly 1, 543-567.
11. DEUTSCH, S. J., SIMS, L. A. and PFEIFER, P. E. (1979). Identification and modeling of dynamic intervention effects associated with public policy decisions. Proceedings of the 2nd International Conference on Mathematical Modeling, Vol. 1, 405-414.
12. ALT, F. B., DEUTSCH, S. J., and GOODE, J. J. (1977). Estimation for the multi-consequence intervention model. Proceedings of Statistical Computing Section, American Statistical Association, 102-107.
13. ALT, F. B., DEUTSCH, S. J. and WALKER, J. (1977). Control Charts for multivariate correlated observations. 31st Annual Technical Conference, American Society for Quality Control.
14. ALT, F. B. and DEUTSCH, S. J. (1978). Multivariate control charts for the mean. Proceedings of the N. E. American Institute of Decision Sciences, Washington, D. C.
15. PFEIFER, P. E. and DEUTSCH, S. J. (1980). A three-stage iterative procedure for space-time modeling. Technometrics 22, No. 1, 35-47.
16. PFEIFER, P. E. and DEUTSCH, S. J. (1980). A comparison of Estimation procedures for the parameters of the STAR model. Communications in Statistics B9, No. 3, 255-270.

17. PFEIFER, P. E. and DEUTSCH, S. J. (1980). Identification and interpretation of first-order space-time ARMA models. Technometrics 22, No. 3, 397-408.

18. PFEIFER, P. E. and DEUTSCH. S. J. (1980). A STARIMA model building procedure with application to description and regional forecasting. Transactions of the Institute of British Geographers 5.

19. PFEIFER, P. E. and DEUTSCH, S. J. (1980). Independence and sphericity tests for the residuals of space-time ARMA models. Communications in Statistics B9, No. 5.

20. PFEIFER, P. E. and DEUTSCH, S. J. (1980). Variance of the sample space-time autocorrelation function. Journal of the Royal Statistical Society, B 42, No. 3.

21. PFEIFER, P. E. and DEUTSCH S. J. (1980). Stationarity and intibility regions for low-order STARMA models. Communications in Statistics B9, No. 5.

22. PFEIFER, P. E. and DEUTSCH, S. J. (1981). Seasonal space-time ARIMA modeling. Geographical Analysis 13, No. 2.

23. PFEIFER, P. E. and DEUTSCH, S. J. (to appear). Variance of the sample space-time autocorrelation function of contemporaneously correlated variables. SIAM Journal of Applied Mathematics, Part A.

24. PFEIFER, P. E. and DEUTSCH, S. J. (1979). Forecasting of space-time ARIMA models. School of Industrial and Systems Engineering Report Series No. J-79-17, Georgia Institute of Technology, Atlanta, Georgia.

25. DEUTSCH, S. J. and PFEIFER, P. E. (1980). Some alternatives for forecasting multiregional systems. ISyE Report Series No. J-80-04, Georgia Institute of Technology, Atlanta, GA.

26. DEUTSCH, S. J. and WANG, C. C. (1979). Modeling multiconsequence interventions. ISyE Report Series No. J-79-33, Georgia Institute of Technology, Atlanta, GA.

27. DEUTSCH, S. J. and WANG, C. C. (1980). Modeling multiple region interventions. ISyE Report Series No. J-80-12, Georgia Institute of Technology, Atlanta, GA.

END