88422

Rhodes Associates

Criminal Justice Research Series

A COMPARISON OF THE AMEMIYA GLS AND
THE LEE-MADDALA-TROST G2SLS IN A
SIMULTANEOUS-EQUATIONS TOBIT MODEL

by

Takeshi Amemiya

May, 1982

CJRS 3

Prepared under Grant Number 81-IJ-CX-0055 from the National Institute of Justice, Law Enforcement Assistance Administration, U. S. Department of Justice. Points of view or opinions stated in this document are those of the author and do not necessarily represent the official positions or policies of the U. S. Department of Justice.

Rhodes Associates, 706 Cowper Street, 3rd flr, Palo Alto, California 94301 (415) 326 6246

A COMPARISON OF THE AMEMIYA GLS AND
THE LEE-MADDALA-TROST G2SLS IN A
SIMULTANEOUS-EQUATIONS TOBIT MODEL

by

Takeshi Amemiya*

Amemiya [1978 and 1979] proposed a method of obtaining estimates of structural parameters from given estimates of reduced-form parameters in simultaneous-equations probit and Tobit models respectively. In these papers I discussed both LS and GLS estimators, but in this paper I will consider only GLS. It should be noted that in these papers I actually proposed a class of LS and GLS estimators, since different estimators of structural parameters result (even asymptotically) from using different estimators of reduced-form parameters to begin with. Lee, Maddala, and Trost [1980] proposed an alternative method of estimating structural parameters in a simultaneous-equations Tobit model, which I will call the LMT-2SLS estimator. This estimator was generalized by Lee [1981] to take account of a non-scalar covariance matrix and yielded what I will call the LMT-G2SLS estimator. It should be noted that the asymptotic properties of the LMT-2SLS and the LMT-G2SLS estimators do not depend upon the choice of the estimator of the reduced-form parameters, provided that the latter is a consistent estimator. Lee [1981] demonstrated that in a simultaneous-equation Tobit model the LMT-G2SLS estimator is asymptotically more efficient than the Amemiya GLS estimator. In this paper I will point out that the

---

*This research was supported by National Institute of Justice Grant No. 81-IJ-CX-0055 to Rhodes Associates.

Amemiya GLS estimator which Lee found to be inferior is merely a member of the class of the Amemiya GLS estimators and that the Amemiya GLS class actually contains members which beat the LMT-G2SLS as well as one which is asymptotically equivalent to the LMT-G2SLS.

The model I will consider throughout this paper is defined as follows:

$$(1) \qquad y_t = Y_{1t}'\gamma_1 + X_{1t}'\beta_1 + u_t$$

and

$$(2) \qquad Y_t' = X_t'\Pi + V_t' \ , \quad t = 1,2,\dots,T \ ,$$

where Y_t' and X_t' are observed if

$$(3) \qquad w_t > -S_t'\delta \ ,$$

where $Y_t = (y_t, Y_{1t}')'$ is a G-vector of endogenous variables, $X_t = (X_{1t}', X_{2t}')'$ is a $(K_1 + K_2)$-vector of exogenous variables (where I assume $K_2 \geqq G - 1$ for identifiability), γ_1, β_1, and Π are a $(G - 1)$-vector, a K_1-vector, and a $K \times G$ matrix of unknown parameters respectively, and $(u_t, V_t', w_t)'$ is an i.i.d. drawing from a $(G + 2)$-variate normal distribution with zero mean and a general variance-covariance matrix except that $Vw_t = 1$ for normalization. It is assumed that if (3) does not hold we observe that fact and S_t, so that δ, a vector of nuisance parameters, can be consistently estimated by the probit MLE.

We have

$$(4) \qquad E(u_t | w_t > -S_t'\delta) = \mu\lambda(S_t'\delta) \ ,$$

where $\mu = Eu_t w_t$ and $\lambda(\cdot) = \phi(\cdot)/\Phi(\cdot)$ where ϕ and Φ are the density and the distribution function respectively of the standard normal variable, and similarly,

$$E(V_t | w_t > -S_t'\delta) = \theta\lambda(S_t'\delta) \quad , \tag{5}$$

where $\theta = EV_t w_t$. Using (4) and (5), we can rewrite (1) and (2) as

$$y_t = Y_{1t}'\gamma_1 + X_{1t}'\beta_1 + \mu\lambda(S_t'\hat{\delta}) + \bar{u}_t + \mu[\lambda(S_t'\delta) - \lambda(S_t'\hat{\delta})] \tag{6}$$

and

$$Y_t' = X_t'\Pi + \lambda(S_t'\hat{\delta})\theta' + \bar{V}_t' + [\lambda(S_t'\delta) - \lambda(S_t'\hat{\delta})]\theta' \quad , \tag{7}$$

where $\bar{u}_t = u_t - E(u_t | w_t > -S_t'\delta)$, $\bar{V}_t = V_t - E(V_t | w_t > -S_t'\delta)$, and $\hat{\delta}$ is the probit MLE of δ (i.e., $\hat{\delta}$ is the value of δ that maximizes $\prod_1 \Phi(S_t'\delta) \prod_0 [1 - \Phi(S_t'\delta)]$ where $\prod_1$ is the product over those t for which (3) holds and $\prod_0$ is the product over those t for which (3) does not hold).

Now, I will rewrite (6) and (7) in vector notation, but in doing so, I will use only those periods for which (3) holds. Thus, the following vectors and matrices of observations have only T_1 rows, T_1 being the number of periods for which (3) holds:

$$y = Y_1\gamma_1 + X_1\beta_1 + \mu\hat{\lambda} + \bar{u} + \mu(\lambda - \hat{\lambda}) \equiv Z\alpha + \mu\hat{\lambda} + \varepsilon \tag{8}$$

and

$$Y = X\Pi + \hat{\lambda}\theta' + \bar{V} + (\lambda - \hat{\lambda})\theta' \quad . \tag{9}$$

We will assume that $\lim_{T_1 \to \infty} T_1^{-1} X'X$ exists and is positive-definite.

A simple consistent estimator of Π, denoted $\hat{\Pi}$, can be obtained by applying LS to (9):1/

$$\hat{\Pi} = (X'\hat{M}X)^{-1}X'\hat{M}Y \quad , \tag{10}$$

where $\hat{M} = I - (\hat{\lambda}'\hat{\lambda})^{-1}\hat{\lambda}\hat{\lambda}'$.

Given $\hat{\Pi}$, the LMT-G2SLS estimator of $\alpha = (\gamma_1', \beta_1')'$, denoted $\hat{\alpha}_L$, is defined by

$$\hat{\alpha}_L = (\hat{Z}'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}Z)^{-1}\hat{Z}'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}y \quad , \tag{11}$$

where $\hat{Z} = (X\hat{\Pi}, X_1)$, $\hat{M}_1 = I - (\hat{\lambda}'\Sigma^{-1}\hat{\lambda})^{-1}\Sigma^{-\frac{1}{2}}\hat{\lambda}\hat{\lambda}'\Sigma^{-\frac{1}{2}}$, and Σ denotes the asymptotic variance-covariance matrix of ε. In practice, Σ must be estimated, but I will proceed as if Σ were known since all the asymptotic results of the paper remain valid if Σ is replaced by a consistent estimate of Σ. The asymptotic variance covariance matrix of $\hat{\alpha}_L$, denoted $V\hat{\alpha}_L$, is given by

$$V\hat{\alpha}_L = (\bar{Z}'\Sigma^{-\frac{1}{2}}M_1\Sigma^{-\frac{1}{2}}\bar{Z})^{-1} \quad , \tag{12}$$

where $\bar{Z} = (X\Pi, X_1)$ and M_1 is obtained from $\hat{M}_1$ by omitting the ^ over λ. Though I defined $\hat{\alpha}$ using $\hat{\Pi}$ in the above, the asymptotic variance-covariance matrix $V\hat{\alpha}_L$ is unchanged if any other consistent estimator of Π is used.2/

In order to define the Amemiya GLS class, I use the following identity relationship between the structural and reduced-form parameters:

$$(13) \qquad \pi = \Pi_1\gamma_1 + J\beta_1 \quad ,$$

where $\Pi = [\pi, \Pi_1]$ and $J = [I, 0]'$ where I is the identity matrix of size K_1 and 0 is the $K_2 \times K_1$ matrix of zeroes. Let $\tilde{\Pi}$ be an arbitrary consistent estimator of Π such that $\tilde{\Pi} - \Pi$ is of the order of $T_1^{-\frac{1}{2}}$. Then, (13) can be rewritten as

$$(14) \qquad \tilde{\pi} = \tilde{\Pi}_1\gamma_1 + J\beta_1 + \eta \equiv \tilde{A}\alpha + \eta \quad ,$$

where $\eta = \tilde{\pi} - \pi - (\tilde{\Pi}_1 - \Pi_1)\gamma_1$. Then, the Amemiya GLS class, denoted $\tilde{\alpha}_A$, is defined by

$$(15) \qquad \tilde{\alpha}_A = (\tilde{A}'\Omega^{-1}\tilde{A})^{-1}\tilde{A}'\Omega^{-1}\tilde{\pi} \quad ,$$

where Ω denotes the asymptotic variance-covariance matrix of η. In practice, Ω must be consistently estimated, but the asymptotic results are unchanged. Note that (15) defines a class of estimators since we let $\tilde{\Pi}$ vary among all the consistent estimators of the specified order. Note that Ω also changes with the choice of estimator of Π. The asymptotic variance-covariance matrix of $\tilde{\alpha}_A$ is given by

$$(16) \qquad V\tilde{\alpha}_A = (A'\Omega^{-1}A)^{-1} \quad ,$$

where $A = (\Pi_1, J)$.

Let $\hat{\alpha}_A$ be a member of the class (15) obtained by using $\hat{\Pi}$ defined in (10) in place of $\tilde{\Pi}$. Then, Lee [1981] proved

$$(17) \qquad V\hat{\alpha}_L \leqq V\hat{\alpha}_A \quad ,$$

where (17) means that the right-hand side minus the left-hand side is nonnegative definite. I will give a proof in my notation. It is easy to show that $\hat{\alpha}_A$ is the GLS estimator applied to

$$(18) \qquad X'\hat{M}y = X'\hat{M}Z\alpha + X'\hat{M}\varepsilon \quad .$$

Therefore, we have

$$(19) \qquad V\hat{\alpha}_A = [\bar{Z}'MX(X'M\Sigma MX)^{-1}X'M\bar{Z}]^{-1} \quad .$$

Therefore, (17) follows from the matrix inequality

$$(20) \qquad I \geqq (\lambda'\Sigma^{-1}\lambda)^{-1}\Sigma^{-\frac{1}{2}}\lambda\lambda'\Sigma^{-\frac{1}{2}} + \Sigma^{\frac{1}{2}}MX(X'M\Sigma MX)^{-1}X'M\Sigma^{\frac{1}{2}} \quad ,$$

which can be proved by noting that the matrices in the right-hand side of (20) are projection matrices projecting onto the spaces spanned by $\Sigma^{-\frac{1}{2}}\lambda$ and $\Sigma^{\frac{1}{2}}MX$ respectively, which are orthogonal to each other.

Next, I will obtain a member of the Amemiya GLS class which has the same asymptotic variance-covariance matrix as the LMT-G2SLS estimator. Such a member, denoted α_A^*, is obtained by using

$$(21) \qquad \Pi^* = (X'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}X)^{-1}X'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}Y$$

in place of $\tilde{\Pi}$ in the definition (15). It is easy to show that α_A^* is the GLS estimator applied to

$$X'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}y = X'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}Z\alpha + X'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}\varepsilon \tag{22}$$

and hence

$$V\alpha_A^* = V\hat{\alpha}_L \quad . \tag{23}$$

Finally, I will show that the Amemiya GLS class contains estimators which are asymptotically more efficient than the LMT-G2SLS estimator. From (16) it is clear that the smaller (in matrix sense) is Ω, the smaller is $V\tilde{\alpha}_A$. Therefore, the better the estimator of Π one uses in (15), the smaller $V\tilde{\alpha}_A$ becomes. This fact and (23) imply that a member of the Amemiya GLS class which uses an asymptotically more efficient estimator of Π than Π^* beats the LMT-G2SLS. One can find such estimators. I will mention two examples. One is the GLS estimator applied to (9). It is asymptotically better than Π^* since the latter is a "wrong" GLS estimator applied to the same equation. The other is the maximum likelihood estimator of Π in the model defined by (2) and (3). Since this estimator is asymptotically efficient, it is better than either Π^* or the correct GLS estimator mentioned above. However, I should point out that the estimator of α based on either of these two estimators of Π is computationally more burdensome than either α_A^* or $\hat{\alpha}_L$ because in these cases Ω depends on γ_1 and hence one must first obtain a consistent estimate of γ_1 in some way.

Footnotes

1. This estimator was first suggested by Heckman [1976] and its asymptotic variance-covariance matrix is given in Heckman [1979].
2. This can be easily proved by proving plim $T_1^{-1}[\hat{Z}'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}Z - \bar{Z}'\Sigma^{-\frac{1}{2}}M_1\Sigma^{-\frac{1}{2}}\bar{Z}] = 0$ and plim $T_1^{-\frac{1}{2}}[\hat{Z}'\Sigma^{-\frac{1}{2}}\hat{M}_1\Sigma^{-\frac{1}{2}}\varepsilon - \bar{Z}'\Sigma^{-\frac{1}{2}}M_1\Sigma^{-\frac{1}{2}}\varepsilon] = 0$ for an arbitrary consistent estimator $\hat{\Pi}$.

References

Amemiya, T. [1978], "The Estimation of a Simultaneous Equation Generalized Probit Model," Econometrica, Vol. 46, 1193-1205.

Amemiya, T. [1979], "The Estimation of a Simultaneous-Equation Tobit Model," International Economic Review, Vol. 20, 169-181.

Heckman, J.J. [1976], "The Common Structure of Statistical Models of Truncation, Sample Selection, and Limited Dependent Variables and a Simple Estimator for Such Models," Annals of Economic and Social Measurement, Vol. 5, 475-492.

Heckman, J.J. [1979], "Sample Bias as a Specification Error," Econometrica, Vol. 47, 153-162.

Lee, L.F., G.S. Maddala, and R.P. Trost [1980], "Asymptotic Covariance Matrices of Two-Stage Probit and Two-Stage Tobit Methods for Simultaneous Equations Models with Selectivity," Econometrica, Vol. 48, 491-504.

Lee, L.F. [1981], "Simultaneous Equations Models with Discrete and Censored Variables," in C.F. Manski and D. McFadden (eds.) Structural Analysis of Discrete Data with Econometric Applications. MIT Press, Cambridge, Mass.

END